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## Shintani Functions and Rankin-Selberg Convolution

### I. Local Theory

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In this note, we report a recent progress of the local theory of Shintani functions on split orthogonal groups (joint work with Shin-ichi Kato).

#### §1. Notation

Let  $F$  be a non-archimedean local field with  $\text{char}(F) \neq 2$  and denote by  $\mathfrak{o}$  the integer ring of  $F$ . Fix a prime element  $\pi$  of  $F$  and put  $q = \#(\mathfrak{o}/\pi\mathfrak{o})$ . For a positive integer  $n$ , we put

$$S_n = \begin{cases} \begin{bmatrix} 0 & J_v \\ J_v & 0 \end{bmatrix} & \text{if } n \text{ is even} \\ \begin{bmatrix} 0 & 0 & J_v \\ 0 & 2 & 0 \\ J_v & 0 & 0 \end{bmatrix} & \text{if } n \text{ is odd} \end{cases}$$

where  $v = \begin{bmatrix} n \\ 2 \end{bmatrix}$  and  $J_v = \begin{bmatrix} 0 & 1 \\ & \ddots \\ 1 & 0 \end{bmatrix} \in \text{GL}_v(F)$ . Let  $G_n$  be the orthogonal group of  $S_n$  over  $F$ :  $G_n = \{g \in \text{GL}_n(F) \mid {}^t g S_n g = S_n\}$ . We define an embedding  $\iota_n$  of  $G_{n-1}$  into  $G_n$  as follows (we put  $v' = \begin{bmatrix} n-1 \\ 2 \end{bmatrix}$ ):

(a) If  $n$  is even,

$$\iota_n \left( \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \right) = \begin{bmatrix} a_1 & \frac{a_2}{2} & \frac{a_2}{2} & a_3 \\ b_1 & \frac{b_2+1}{2} & \frac{b_2-1}{2} & b_3 \\ b_1 & \frac{b_2-1}{2} & \frac{b_2+1}{2} & b_3 \\ c_1 & \frac{c_2}{2} & \frac{c_2}{2} & c_3 \end{bmatrix}$$

where  $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \in G_{n-1}$  is the block decomposition according to the partition  $n-1 = v' + 1 + v'$ .

(b) If  $n$  is odd,

$$\iota_n \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{bmatrix}$$

where  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G_{n-1}$  is the block decomposition according to the partition  $n-1 = v' + v'$ .

In what follows, we write  $G$  and  $G'$  for  $G_n$  and  $G_{n-1}$  respectively. Let  $v = [n/2]$  (resp.  $v' = [(n-1)/2]$ ) be the Witt index of  $S_n$  (resp.  $S_{n-1}$ ). We identify  $G'$  with a subgroup of  $G$  via  $\iota_n$ . Put  $K = G \cap GL_n(o)$  (resp.  $K' = G' \cap GL_{n-1}(o)$ ), and let  $\mathcal{H} = \mathcal{H}(G, K)$  (resp.  $\mathcal{H}' = \mathcal{H}(G', K')$ ) be the Hecke algebra of  $(G, K)$  (resp.  $(G', K')$ ). To parametrize  $\text{Hom}_{\mathbb{C}}(\mathcal{H}, \mathbb{C})$ , let  $X_{\text{unr}}(F^\times)^v$  be the group of  $v$ -tuples of unramified characters of  $F^\times$ . Let  $P$  be the subgroup of upper triangular matrices in  $G$  (the standard minimal parabolic subgroup of  $G$ ). Then  $\chi \in X_{\text{unr}}(F^\times)^v$  is regarded as a character of  $P$  in a natural manner. Define a function  $\Phi_\chi$  on  $G$  to be  $\Phi_\chi(pk) = (\chi \delta_P^{1/2})(p)$  for  $p \in P$  and  $k \in K$ , where  $\delta_P$  is the module of  $P$ . For  $\varphi \in \mathcal{H}$ , put

$$\chi^\wedge(\varphi) = \int_G \varphi(g) \Phi_\chi(g) dg.$$

Then  $\varphi \mapsto \chi^\wedge(\varphi)$  defines an element of  $\text{Hom}_{\mathbb{C}}(\mathcal{H}, \mathbb{C})$ . The correspondence  $\chi \mapsto \chi^\wedge$  gives rise to a bijection from  $X_{\text{unr}}(F^\times)^\vee / W_G$  onto  $\text{Hom}_{\mathbb{C}}(\mathcal{H}, \mathbb{C})$ , where  $W_G$  is the Weyl group of  $G$  (cf. [Sa]). Similarly we can identify  $\text{Hom}_{\mathbb{C}}(\mathcal{H}', \mathbb{C})$  with  $X_{\text{unr}}(F^\times)^\vee / W_{G'}$ , where  $W_{G'}$  is the Weyl group of  $G'$ .

## §2. Main results

As in [MS1], we define the space  $\text{Sh}(\chi, \chi')$  of local Shintani functions on  $G$  attached to  $(\chi, \chi') \in X_{\text{unr}}(F^\times)^\vee \times X_{\text{unr}}(F^\times)^\vee$  by

$$\begin{aligned} \text{Sh}(\chi, \chi') = \{ W: G \rightarrow \mathbb{C} \mid & \text{(i) } W(k'gk) = W(g) \text{ (} k' \in K', k \in K, g \in G) \\ & \text{(ii) } \varphi' * W * \varphi = \chi'^\wedge(\varphi') \chi^\wedge(\varphi) W \text{ (} \varphi' \in \mathcal{H}', \varphi \in \mathcal{H} \text{)} \}. \end{aligned}$$

Here we put

$$(\varphi' * W * \varphi)(g) = \int_{G'} dx' \int_G dx \varphi'(x') W(x'^{-1} g x) \varphi(x).$$

Note that Shintani functions can be regarded as spherical functions on a spherical homogeneous space  $X = G^{\text{diag}} \backslash G' \times G$ , where  $G^{\text{diag}} = \{ (g', g') \mid g' \in G' \}$  is a spherical subgroup of  $G' \times G$  in the sense of [Br]. The following has been conjectured in [MS1].

**Theorem 1** *Let  $(\chi, \chi') \in X_{\text{unr}}(F^\times)^\vee \times X_{\text{unr}}(F^\times)^\vee$ . Then we have  $\dim_{\mathbb{C}} \text{Sh}(\chi, \chi') = 1$ . Moreover, there exists a  $W_{\chi, \chi'} \in \text{Sh}(\chi, \chi')$  with  $W_{\chi, \chi'}(1) = 1$ .*

To state an explicit formula for  $W_{\chi, \chi'}$ , we need several preparations.

Let  $\Lambda_v = \{ (m_1, \dots, m_v) \in \mathbb{Z}^v \mid m_1 \geq \dots \geq m_v \geq 0 \}$ . For  $m = (m_1, \dots, m_v) \in \Lambda_v$ ,

we put  $\Pi_m = d_n \left( \begin{bmatrix} \pi^{m_1} & & 0 \\ & \ddots & \\ 0 & & \pi^{m_v} \end{bmatrix} \right) \in G$ , where

$$d_n(A) = \begin{cases} \begin{bmatrix} A & 0 \\ 0 & J_v {}^t A^{-1} J_v \end{bmatrix} & \text{if } n \text{ is even} \\ \begin{bmatrix} A & 0 \\ & 1 \\ 0 & J_v {}^t A^{-1} J_v \end{bmatrix} & \text{if } n \text{ is odd} \end{cases} \quad \text{for } A \in GL_v(F).$$

Similarly we define  $\Pi'_{m'} \in G'$  for  $m' \in \Lambda_{v'}$ . Let  $g_0$  be an element of  $G$  given by

$$g_0 = \begin{cases} d_n(A_0) & \text{if } n \text{ is even} \\ \begin{bmatrix} 1_v & -2\eta & -\eta {}^t \eta J_v \\ 0 & 1 & {}^t \eta J_v \\ 0 & 0 & 1_v \end{bmatrix} & \text{if } n \text{ is odd} \end{cases}$$

where  $A_0 = \begin{bmatrix} 1 & 0 & 1 \\ & \ddots & \vdots \\ 0 & 1 & 1 \\ 0 & \dots & 0 & 1 \end{bmatrix} \in GL_v(F)$  and  $\eta = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in F^v$ . The following result is

the "Cartan decomposition" for  $\mathcal{X}$ .

**Proposition 2** *We have*

$$G = \coprod_{m \in \Lambda_v, m' \in \Lambda_{v'}} K' g(m, m') K \quad (\text{disjoint union})$$

where  $g(m, m') = \Pi'_{m'} g_0 \Pi_m \in G$ .

Put

$$Q_n(q) = \prod_{i=1}^{v-1} (1 - q^{-2i}) \times \begin{cases} 1 & \text{if } n \text{ is even} \\ (1 - q^{-v}) & \text{if } n \text{ is odd.} \end{cases}$$

Let  $\chi = (\chi_1, \dots, \chi_v) \in X_{\text{unr}}(F^\times)^v$  and  $\chi' = (\chi'_1, \dots, \chi'_{v'}) \in X_{\text{unr}}(F^\times)^{v'}$ . To simplify notation, we often write  $\chi'_i$  and  $\chi_j$  for the values  $\chi'_i(\pi)$  and  $\chi_j(\pi)$  respectively. Define

$$\mathcal{D}(\chi, \chi') = \frac{\prod_{1 \leq i \leq v', 1 \leq j \leq v} (1 - q^{-1/2} (\chi_i^{-1} \chi_j)^{\varepsilon_{ij}}) (1 - q^{-1/2} (\chi'_i \chi_j)^{-1})}{\Delta_G(\chi) \cdot \Delta_{G'}(\chi')}$$

where

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } i < j \\ -1 & \text{if } i \geq j \end{cases}$$

and

$$\Delta_G(\chi) = \prod_{1 \leq k \leq \ell \leq v} (1 - \chi_k^{-1} \chi_\ell) (1 - \chi_k^{-1} \chi_\ell^{-1}) \times \begin{cases} 1 & \text{if } n \text{ is even} \\ \prod_{1 \leq k \leq v} (1 - \chi_k^{-2}) & \text{if } n \text{ is odd} \end{cases}$$

( $\Delta_{G'}(\chi')$  is similarly defined).

**Theorem 3** Let  $W_{\chi, \chi'} \in \text{Sh}(\chi, \chi')$  be as in Theorem 1. Then, for  $(m, m') \in \Lambda_v \times \Lambda_{v'}$ , we have

$$\begin{aligned} & W_{\chi, \chi'}(g(m, m')) \\ &= \frac{1}{Q_n(q)} \sum_{\substack{w \in W_G \\ w' \in W_{G'}}} \mathcal{D}(w\chi, w'\chi') (w\chi \cdot \delta_P^{1/2})(\Pi_m) (w'\chi' \cdot \delta_{P'}^{1/2})(\Pi_{m'}), \end{aligned}$$

where  $W_G$  (resp.  $W_{G'}$ ) is the Weyl group of  $G$  (resp.  $G'$ ) and  $\delta_P$  (resp.  $\delta_{P'}$ ) is the module of the standard minimal parabolic subgroup  $P$  (resp.  $P'$ ) of  $G$  (resp.  $G'$ ).

### §3. Sketch of proof

The existence part of Theorem 1 is proved by using an integral expression of Shintani functions similar to that of [MS2]. We can prove Theorem 3 following the method of [KM], where an explicit formula for local Shintani functions on  $GL(n)$  is shown.

We now give an outline of the proof of the uniqueness part of Theorem

1. For  $(m, m') \in \Lambda = \Lambda_v \times \Lambda_{v'}$ , we define an element  $\{m, m'\}$  of  $Z^{v+v'}$  by

$$\{m, m'\} = \begin{cases} (m_1, m'_1, m_2, m'_2, \dots, m_{v'}, m'_{v'}, m_v) & \text{if } n \text{ is even (in this case } v = v' + 1) \\ (m_1, m'_1, m_2, m'_2, \dots, m_{v'}, m'_{v'}) & \text{if } n \text{ is odd (in this case } v = v'). \end{cases}$$

We define a total ordering of  $\Lambda$  as follows:  $(\ell, \ell') \prec (m, m')$  if and only if  $\{\ell, \ell'\} < \{m, m'\}$  (in the usual lexicographic ordering of  $\mathbb{Z}^{v+v'}$ ). The proof of the uniqueness of Shintani functions is reduced to the following:

**Proposition 4** *Let  $W \in \text{Sh}(\chi, \chi')$  and  $(m, m') \in \Lambda$ . Then we have*

$$W(g(m, m')) = \sum c_{\ell, \ell'}(\chi, \chi') W(g(\ell, \ell')),$$

where the summation is over  $(\ell, \ell') \in \Lambda$  with  $(\ell, \ell') \prec (m, m')$ , and  $c_{\ell, \ell'}(\chi, \chi')$  is an element of  $\mathbb{C}[\chi_1^{\pm 1}, \dots, \chi_v^{\pm 1}, (\chi'_1)^{\pm 1}, \dots, (\chi'_{v'})^{\pm 1}]$  depending only on  $(\ell, \ell')$  and  $(\chi, \chi')$  and not on  $W$ .

The proposition follows from the next result, which is an analogue of Proposition (4.4.4) in [BT].

**Key lemma** *Let  $(m, m'), (\ell, \ell') \in \Lambda$  and  $k \in K$ , and suppose that  $\Pi'_m k \Pi_m \in K' g(\ell, \ell') K$ . Then we have  $(\ell, \ell') \preceq (m, m')$ .*

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